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Schwinger Terms and Cohomology of Pseudodifferential Operators

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Abstract

We study the cohomology of the Schwinger term arising in second quantization of the class of observables belonging to the restricted general linear algebra. We prove that, for all pseudodifferential operators in 3+1 dimensions of this type, the Schwinger term is equivalent to the "twisted" Radul cocycle, a modified version of the Radul cocycle arising in non-commutative differential geometry. In the process we also show how the ordinary Radul cocycle for any pair of pseudodifferential operators in any dimension can be written as the phase space integral of the star commutator of their symbols projected to the appropriate asymptotic component.

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1. Introduction

Current algebras play an important role in many quantum field theories. Historically, they were introduced in an attempt to describe hadronic processes. The hope was that the relevant physics would be captured by a restricted set of operators, the currents, satisfying linear commutation relations among themselves, and by a hamiltonian, bilinear in the currents, describing their time evolution. Even after the advent of QCD as the "microscopic" theory of strong interactions, physicists have often used current algebra techniques in the kinematical regions where the fundamental theory becomes intractable.

When seen from the point of view of a more fundamental theory, the currents are interpreted as composite operators in terms of the elementary fields, e.g. bilinears in some fermionic matter field. Often, at the quantum level, the naive conservation laws and commutation relations of the currents have to be modified by the addition of extra terms. In particular, when they spoil the conservation laws of some classically conserved current, these terms are referred to as anomalies. These anomalies are of crucial importance for the physical applications of the algebra; for global algebras they are known to determine e.g. the decay rate of π mesons, while for local algebras one is confronted by unitarity problems if the extension cannot be eliminated by choosing the particle content of the theory properly.

When appearing directly in the current–current commutation relations, these terms are also referred to as Schwinger terms because originally such terms were introduced by Schwinger in the context of QED [Sc-59]. From the point of view of the fundamental theory, they should be generated by the regularization procedure needed to make the current a well-defined composite operator. Their effect on the commutation relations can be understood in terms of Lie algebra cohomology as giving a certain central or abelian (perhaps even non-abelian) extension of the naive current algebra.

We will only consider the case where the currents are bilinear in some fermionic field. In 1+1 dimensions we know from many thoroughly studied examples (e.g. affine Kac–Moody algebras [Ba-71, Ka-67, Mo-67]) that normal ordering suffices to make such currents well defined and that, in general, central extensions are generated. In 3+1 and higher dimensions the situation changes dramatically in that normal ordering alone is not enough to render the bilinear expressions for the currents well defined. However, although in perturbation theory a (wave function) renormalization that successfully eliminates this problem can be implemented, it is still not understood how to define the currents in a completely regular fashion.

As we will discuss extensively below, some of these concepts can be rigorously formulated using the language of second quantization. In particular, to any observable in the one-particle Hilbert space, one can associate a fermionic bilinear acting in some Fock space. From this point of view, the ordinary currents are thought of as second-quantized multiplicative operators, and in dimensions higher than 1+1 they require further regularization in addition to normal ordering.

It is of interest to isolate the observables for which normal ordering is sufficient even in higher dimensions. These form what is known as the restricted general linear algebra $\mathfrak{gl}_{\mathrm{res}}$ of the one-particle Hilbert space. In particular, we will show that it is possible to characterize these operators

explicitly by considering only pseudodifferential operators (ΨDOs). This can hardly be regarded as a loss of generality, since all the operators of interest in physics can be regarded as ΨDOs of some kind. The real restriction is in considering only operators in $\mathfrak{gl}_{\text{res}}$. Nevertheless, the study of $\mathfrak{gl}_{\text{res}}$ is of great interest for at least three independent reasons:

i) The approach works in (1+1)-dimensional spacetime, in the sense that normal ordering in this case suffices to regularize most operators. In particular, all affine Kac–Moody algebras can be understood in this way.

ii) In higher dimensions $\mathfrak{gl}_{\text{res}}$ represents a simple subclass of operators that can be studied very explicitly, still displaying non-trivial properties such as the presence of Schwinger terms in their commutators. Any future understanding of representation theory of higher-dimensional current algebras must eventually agree with the results obtained for this subclass.

iii) $\mathfrak{gl}_{\text{res}}$ may actually be of crucial importance in developing the representation theory mentioned above. It has recently been proposed by Mickelsson [Mi-93] that the elements of $\mathfrak{gl}_{\text{res}}$ should be used as regularized versions of the more singular operators one is actually interested in. $\mathfrak{gl}_{\text{res}}$ should play a similar role in the study of the generalization of higher-dimensional current algebras recently discovered in [Ce-94, Fe-94].

Normal ordering of the second quantized ΨDOs in $\mathfrak{gl}_{\text{res}}$ generates Schwinger terms which appear as two-cocycles of the underlying Lie algebra. As such, they define a central extension $\widehat{\mathfrak{gl}}_{\text{res}}$ of $\mathfrak{gl}_{\text{res}}$. However, when discussing ΨDOs one finds that the requirement of making them smooth at zero momentum introduces a regulating function, i.e. the Schwinger term becomes regularization dependent. This is an unwanted feature of the procedure, and it is crucial to find a way to extract the cohomological information, or, in other words, to relate the cohomology class of the Schwinger term to one of the known cohomologies in the space of ΨDOs. How this can be done is one of the two main results of our paper:

- *The Schwinger term for* ΨDO*'s in* $\mathfrak{gl}_{\text{res}}$ *lies in the same cohomology class as the so-called "twisted" Radul cocycle* [Mi-94]*, a slightly modified version of the well-known Radul cocycle used in non-commutative differential geometry.*

Our second main result (that will actually be proven first) is not related in any way to the structure of $\mathfrak{gl}_{\text{res}}$, but is a general result on the cohomology of ΨDOs:

- *In any number of dimensions n, the Radul cocycle of two arbitrary* ΨDO*'s (not necessarily in* $\mathfrak{gl}_{\text{res}}$*) can be written as the integral over all phase space of their commutator projected onto the component with asymptotic behavior $|p|^{-n}$.*

The paper is organized as follows. After some introductory material on second quantization and Schwinger terms in sections 2 and 3, respectively, we introduce ΨDOs in sect. 4. These short sections cover only well-known material and are added primarily in an attempt to make the paper easier to read and to a certain extent self-contained. In sect. 5 we prove that the Radul cocycle can be expressed as a commutator as stated above. In sect. 6 we characterize the ΨDOs that belong to $\mathfrak{gl}_{\text{res}}$ and use this characterization in sect. 7 to relate the Radul cocycle to the Schwinger term. Some additional remarks are added in sect. 8 and we mention a few cases where our results are directly relevant, namely, affine Kac–Moody algebras [Ba-71, Ka-67, Mo-67] in 1+1 dimensions, Mickelsson–Faddeev–Shatashvili algebras in 3+1 dimensions [Fa-84a, Fa-84b, Mi-83] and a recently

proposed extension of the algebra of maps from an n-dimensional manifold into a semisimple Lie algebra [Ce-94, Fe-94]. We plan to return to these examples, particularly the last one, in a future publication. For some recent results in this area, see [Ba-93, Ca-94, La-94a].

2. From first to second quantization

Consider a particle moving in Minkowski space $\mathbb{R}^{(n,1)}$. In quantum mechanics, the dynamics of such a particle is specified by giving the time evolution of its wave function $\Psi : \mathbb{R}^n \to V$ up to an overall complex phase. Here V denotes the M-dimensional complex vector space describing the other degrees of freedom of the particle, namely spin and color. Throughout this paper we will only consider the case of half-integral spin. "Color" here simply means any internal symmetry the system may have.

To be specific, we will in sect. 6 restrict our attention to (3+1)-dimensional Weyl spinors transforming in the fundamental representation of the color group $\mathfrak{su}(N)$. The wave function Ψ is then valued in the $(M = 2N)$–dimensional complex vector space $V = \mathbb{C}^2_{\text{spin}} \otimes \mathbb{C}^N_{\text{color}}$. The restriction to Weyl spinors in 3+1 dimensions will be made because, on the one hand, this is the most interesting case due to its direct connection to chiral gauge theory and, on the other hand, it is simple enough to allow explicit calculations, yet general enough to display all the issues we want to discuss. However, the analysis can be repeated for particles with other spins [La-91]. In any case, all that is said in this and in the following section depends only on the fermionic nature of the matter field and not on the specific representation or spacetime dimension.

We must of course restrict ourselves to square-integrable wave functions forming the first-quantized Hilbert space $\mathscr{H}$. At the level of quantum mechanics, the observables are described by self-adjoint operators $A = A^\dagger$ acting on $\mathscr{H}$. We do not need to worry about questions of domain in the first-quantized Hilbert space since all operators of interest to us are bounded.

However, as is well known from the early days of quantum mechanics this picture is inadequate if we want to describe the relativistic dynamics of elementary particles because the energy E of the free particle is not bounded below and creation/annihilation processes cannot be described. In mathematical terms, $\mathscr{H}$ carries a representation of the algebra of observables (to which E belongs) that is not highest (actually lowest) weight. The solution to this problem in the Hamiltonian formulation is also well known; precisely because of its privileged status in defining the lowest weight, one uses the first-quantized energy operator E to define a polarization, i.e. a splitting of the Hilbert space into non-negative and negative energy spaces $\mathscr{H} = \mathscr{H}_+ \oplus \mathscr{H}_-$. Then one introduces a new Hilbert space $\mathscr{F}$ (the Fock space), a lowest weight vector $|0\rangle \in \mathscr{F}$ (the vacuum), and a set of operators acting on $\mathscr{F}$, $a(\Psi)$ and $a^\dagger(\Psi)$ (the annihilation and creation operators, respectively), satisfying $a(\Psi)|0\rangle = 0$ if $\Psi \in \mathscr{H}_+$ and $a^\dagger(\Psi)|0\rangle = 0$ if $\Psi \in \mathscr{H}_-$. Since we are only considering fermionic fields, the spin–statistics theorem requires that these operators satisfy the anti-commutation relations $\{a(\Psi_1), a^\dagger(\Psi_2)\} = \langle\Psi_1|\Psi_2\rangle$.

With these assumptions, the Fock space carries an irreducible representation of the canonical anticommutation relations. One can then represent the algebra of observables in the Fock space, i.e. second quantize the theory, as follows. Consider a basis of eigenfunctions $\{\psi_n\} \in \mathscr{H}$ of E. Here

n is a generic multi-index labeling the elements of the basis and we write symbolically $n \geq 0$ iff $\psi_n \in \mathscr{H}_+$ and $n < 0$ iff $\psi_n \in \mathscr{H}_-$. Also, for the sake of brevity, we define $a_n = a(\psi_n)$, $a_n^\dagger = a^\dagger(\psi_n)$ and $A_{mn} = \langle \psi_m | A | \psi_n \rangle$. The representation of A in the second-quantized Fock space is then given by the operator $\hat{A} = \sum_{mn} A_{mn} : a_m^\dagger a_n :$, where the colons represent the normal ordering necessary to ensure that the operators have zero vacuum expectation value. One way to realize the normal ordering is by setting

$$: a_m^\dagger a_n : = \begin{cases} -a_n a_m^\dagger & \text{for } n \text{ and } m < 0, \\ a_m^\dagger a_n & \text{otherwise.} \end{cases} \tag{1}$$

If such a representation exists, it is manifestly unitary and lowest weight, i.e. $\hat{E}$ is bounded below by the vacuum energy $\hat{E}|0\rangle = 0$.

What can go wrong in going from first to second quantization? In other words, how do we make sure that $\hat{A}$ exists? The condition to check is that $\hat{A}$ creates states of finite norm out of the vacuum, i.e. that $\|\hat{A}|0\rangle\| < \infty$. This norm can be computed explicitly as

$$\|\hat{A}|0\rangle\|^2 = \langle 0|\hat{A}^\dagger \hat{A}|0\rangle = \sum_{m\geq 0, n<0} A_{mn}^* A_{mn} = \frac{1}{8}\mathrm{Tr}([\mathrm{sign}(E), A]^\dagger [\mathrm{sign}(E), A]), \tag{2}$$

where $\mathrm{sign}(E) = \pm 1$ on $\mathscr{H}_\pm$. Hence, $\hat{A}$ is well defined iff the square of $[\mathrm{sign}(E), A]$ has finite trace in $\mathscr{H}$. Operators whose square have finite trace are known as Hilbert–Schmidt (HS) operators. With respect to the polarization $\mathscr{H} = \mathscr{H}_+ \oplus \mathscr{H}_-$ an arbitrary operator A and, in particular, the sign of the energy operator can be written as

$$A = \begin{pmatrix} A_{++} & A_{+-} \\ A_{-+} & A_{--} \end{pmatrix}, \qquad \mathrm{sign}(E) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{3}$$

Requiring the commutator $[\mathrm{sign}(E), A]$ to be HS is equivalent to requiring that the off-diagonal blocks of A be separately HS. In order for the algebra of observables to close under this property, one must also require that the elements be bounded operators. This algebra is called the restricted general linear algebra $\mathfrak{gl}_{\mathrm{res}}$:

$$\mathfrak{gl}_{\mathrm{res}} = \big\{ A : \mathscr{H} \to \mathscr{H} \ \text{ bounded } \mid [\mathrm{sign}(E), A] \text{ Hilbert–Schmidt} \big\}. \tag{4}$$

At this point, we would like to make a short digression on the precise definition of the trace in order to avoid confusion. Similar comments can be found in [La-94b]. A arbitrary bounded linear operator S on a Hilbert space $\mathscr{H}$ is said to be trace class (see e.g. [Pr-86, Si-79]) if its action on an arbitrary vector $\Phi \in \mathscr{H}$ can be written as

$$S\Phi = \sum_k \lambda_k \langle \psi_k | \Phi \rangle \chi_k, \tag{5}$$

where $\{\psi_n\}$ and $\{\chi_m\}$ are two orthonormal Hilbert bases of $\mathscr{H}$ and $\sum |\lambda_n| < \infty$. (Notice that it may not be possible to choose $\psi_n = \chi_n$ if S does not have a complete set of eigenvalues.) For trace class operators the trace is defined to be

$$\mathrm{Tr}(S) = \sum_k \lambda_k \langle \psi_k | \chi_k \rangle. \tag{6}$$

This series is obviously absolutely convergent since $|\langle\psi_k|\chi_k\rangle| \leq 1$. Such a trace is basis independent; the two families $\{\psi_n\}$ and $\{\chi_m\}$ define the operator, not the trace.

In our applications, however, the Hilbert space comes with a polarization and the kind of trace that we need is

$$\mathrm{Tr}_C S = \mathrm{Tr}\begin{pmatrix} S_{++} & 0 \\ 0 & S_{--} \end{pmatrix} = \frac{1}{2}\mathrm{Tr}(S + \mathrm{sign}(E)\, S\, \mathrm{sign}(E)), \tag{7}$$

with the traces in the middle and on right hand side defined as in (6). By considering $S = [\mathrm{sign}(E), A]^\dagger[\mathrm{sign}(E), A]$, where $A \in \mathfrak{gl}_{\mathrm{res}}$, we see that we could use Tr_C instead of Tr in eq. (2). Clearly, if S is trace class the two definitions coincide. However, the trace Tr_C is convergent for a larger class of operators (called "conditionally trace class" in [La-94b]) since the combination $S{+}\mathrm{sign}(E)S\ \mathrm{sign}(E)$ projects out the potentially too singular off-diagonal terms. The price one has to pay is that the definition of Tr_C depends on the choice of polarization. Obviously, the projection $S \mapsto (1/2)(S + \mathrm{sign}(E)S\mathrm{sign}(E))$ is idempotent, and therefore, whereas $\mathrm{Tr}(S) = (1/2)\mathrm{Tr}(S + \mathrm{sign}(E)S\mathrm{sign}(E))$ only for truly trace class operators, $\mathrm{Tr}_C(S) = (1/2)\mathrm{Tr}_C(S + \mathrm{sign}(E)S\mathrm{sign}(E))$ holds for the whole class of conditionally trace class operators.

Unfortunately, the operators of ordinary quantum mechanics, in general, do not admit a second-quantized representation like the one described above, i.e. they do not belong to $\mathfrak{gl}_{\mathrm{res}}$, and one therefore needs to renormalize the vacuum expectation values [Mi-88, Fu-90, Pi-87, Pi-89]. As an illustrative example, consider a smooth function $f(x)$ with compact support and define the multiplicative operator $(F\Psi)(x) \equiv f(x)\Psi(x)$. It is readily checked that $\hat{F}|0\rangle$ has finite norm, i.e. belongs to $\mathfrak{gl}_{\mathrm{res}}$, only in 1+1 dimensions. Nevertheless, as mentioned in the introduction, there are many reasons for looking at $\mathfrak{gl}_{\mathrm{res}}$, perhaps the most important one being that this allows one to obtain rigorous results for a specific class of observables that will eventually have to be matched by any more general method.

3. The Schwinger term as a two-cocycle

As mentioned in the introduction, one of the subtleties arising in quantum field theory is the appearance of c-number terms in the commutation relations of various operators, so-called Schwinger terms. A simple example of such a term is the one present in the commutator between the space and time components of the normal-ordered electromagnetic current $J_\mu(\vec{x}, t)$ (for, say, QED). The naive expectation that $[J_0(\vec{x}, t), J_k(\vec{y}, t)] = 0$ is frustrated by the fact that current conservation would then require J_0 to vanish. Schwinger postulated the appearance of the derivative of a δ-function on the right hand side of the equation, that, vanishing upon integration, does not spoil the definition of electric charge: $[J_0(\vec{x}, t), J_k(\vec{y}, t)] = \mathrm{const} \times i\partial_k\delta(\vec{x} - \vec{y})$. That this term actually arises can be proven rigorously in 1+1 dimensions by taking the current to be a normal-ordered fermionic bilinear and using point-splitting regularization.

The advantage of restricting ourselves to the operators in $\mathfrak{gl}_{\mathrm{res}}$ is that the same rigorous calculations can be straightforwardly generalized to arbitrary dimensions, if only for a very restricted class of operators. In fact, at this abstract level, nothing depends on the dimension of spacetime,

i.e. on the particular choice of $\mathscr{H}$. Let us thus consider $A, B \in \mathfrak{gl}_{\rm res}$ and set

$$[\hat{A}, \hat{B}] = \widehat{[A, B]} - \frac{1}{2} c_S(A, B). \tag{8}$$

(The factor $-1/2$ is inserted for later convenience.) By taking the vacuum expectation value of both sides, and using the fact that $\langle 0|0\rangle = 1$ and that $\langle 0|\widehat{[A,B]}|0\rangle = 0$ we obtain the Schwinger term

$$\begin{aligned} c_S(A, B) =& - 2\langle 0|[\hat{A}, \hat{B}]|0\rangle \\ =& - \frac{1}{4}\mathrm{Tr}\Big(\mathrm{sign}(E)\big[[\mathrm{sign}(E), A], [\mathrm{sign}(E), B]\big]\Big) \\ =& - \frac{1}{2}\mathrm{Tr}\Big(\mathrm{sign}(E)[\mathrm{sign}(E), A][\mathrm{sign}(E), B]\Big) \\ =& \mathrm{Tr}_C\big([\mathrm{sign}(E), A]B\big). \end{aligned} \tag{9}$$

The traces are convergent precisely because of the HS property that we have assumed for the operators A and B. Moreover, the Schwinger term in (9) turns out to be a two-cocycle of the algebra $\mathfrak{gl}_{\rm res}$ defining a non-trivial central extension known as $\widehat{\mathfrak{gl}_{\rm res}}$.

Let us at this point recall some basic elements of Lie algebra cohomology in order to keep our discussion self-contained. For an extensive discussion of Lie algebra cohomology and its relation to quantum field theory we refer the reader to e.g. [Ka-90, Ki-76, Mi-89]. Given an abstract Lie algebra $\mathscr{L}$, an n-cochain with values in $\mathbb{C}$ is defined as an anti-symmetric n-linear map $c^n : \mathscr{L} \wedge \mathscr{L} \wedge .. \wedge \mathscr{L} \to \mathbb{C}$. We denote the vector space of such n-cochains by $C^n = C^n(\mathscr{L}, \mathbb{C})$. The coboundary operator $\delta : C^n \to C^{n+1}$ is defined by

$$\delta c^n(x_1, x_2, ..., x_{n+1}) = \sum_{i<j} (-1)^{i+j+1} c^n([x_i, x_j], x_1, ..., \hat{x}_i, ..., \hat{x}_j, ..., x_{n+1}), \tag{10}$$

where a caret indicates an absent argument. In particular,

$$\begin{aligned} \delta c^1(x_1, x_2) &= c^1([x_1, x_2]), \\ \delta c^2(x_1, x_2, x_3) &= c^2([x_1, x_2], x_3) + c^2([x_1, x_3], x_2) - c^2([x_2, x_3], x_1). \end{aligned} \tag{11}$$

The basic property of δ is its nilpotency, i.e $\delta^2 = 0$. Cochains such that $\delta c = 0$ are called cocycles (or closed cochains), and cocycles of the form $c = \delta\lambda$ are called coboundaries (or exact cochains). The abelian groups obtained by considering linear combinations of cocycles modulo coboundaries define the Lie algebra cohomology of $\mathscr{L}$. The only application of Lie algebra cohomology that we will need in this paper is that the second cohomology group $H^2(\mathscr{L}, \mathbb{C})$ describes the possible central extensions of $\mathscr{L}$. Namely, on the vector space $\mathscr{L} \oplus \mathbb{C}$ the commutator

$$[(x, \xi), (y, \eta)] = ([x, y], c(x, y)) \tag{12}$$

defines a Lie algebra $\hat{\mathscr{L}}$ (i.e. satisfies the Jacobi identities) if and only if $c(x, y)$ is a two-cocycle. Furthermore, two two-cocycles define isomorphic Lie algebras if their difference is a coboundary. An algebra $\hat{\mathscr{L}}$ obtained in this way, a central extension of $\mathscr{L}$ by $\mathbb{C}$, is thus specified by an element of the second Lie algebra cohomology group $H^2(\mathscr{L}, \mathbb{C})$.

Comparing with the definitions above, it is easily checked that the Schwinger term (9) is in fact a non-trivial two-cocycle sometimes also referred to as the Lundberg cocycle [Lu-76]. Understanding the explicit form of such terms and their relation with other kinds of cohomologies, namely those that arise in the study of pseudodifferential operators (ΨDOs), will be the scope of most of the remainder of this paper.

4. Basic facts about pseudodifferential operators

In order to keep the paper self-contained we present here some well-known facts about pseudodifferential operators (ΨDOs) that will be needed later on. We only give the basic results without proofs and refer the reader to e.g. [Hö-85, La-89, Ta-81, Va-93] for more detailed discussions.

Consider the Hilbert space $\mathscr{H} = \mathrm{L}^2(\mathbb{R}^n) \otimes \mathbb{C}^M$ of square integrable functions $\psi : \mathbb{R}^n \to \mathbb{C}^M$ of x. The ΨDO S acting on $\mathscr{H}$ is defined by

$$S\psi(x) = \int e^{ix\cdot p} s(x,p)\tilde{\psi}(p)\frac{d^n p}{(2\pi)^n}, \tag{13}$$

where $\tilde{\psi}(p) = \int e^{-ix\cdot p}\psi(x)d^n x$ is the Fourier transform of ψ and $s : \mathbb{R}^n \times \mathbb{R}^n \to \mathfrak{gl}(M,\mathbb{C})$ is a smooth function assumed to have compact support in x and at most polynomial growth in p. The function $s(x,p)$ is called the symbol of S which we write as $\mathrm{sym}\,(S) = s$.

A ΨDO S (or its symbol s) is said to be of order m, written as $\mathrm{ord}\,(S) = m$, if it has a leading asymptotic behavior for large $|p|$ of the kind $s(x,p) = \mathscr{O}(|p|^m)$ uniformly in x. Here we will only be concerned with ΨDOs of integral order. A ΨDO whose symbol decreases faster than any power of p is called infinitely smoothing. Two ΨDOs S and R are said to be equivalent if they differ by an infinitely smoothing operator. We will denote such an equivalence by $S \approx R$ for the operators, or by $s \approx r$ for their symbols.

The importance of this equivalence relation is that it allows for the introduction of asymptotic expansions; consider the sequence $\{s_k(x,p), k \leq m\}$, where s_k is a smooth symbol of order k. A symbol s of order m is said to have the asymptotic expansion

$$s(x,p) \approx \sum_{k\leq m} s_k(x,p) \tag{14}$$

if, for each integer $r \leq m$,

$$\mathrm{ord}\left(s(x,p) - \sum_{k=r}^{m} s_k(x,p)\right) = r - 1. \tag{15}$$

It is often most convenient to assume that the symbols s_k in the asymptotic expansion (14) are homogeneous of degree k in p for $|p| > \delta$ and smooth everywhere:

$$s_k(x,\lambda p) = \lambda^k s_k(x,p) \quad \text{for} \quad \lambda > 1, \quad \text{and} \quad |p| \geq \delta > 0. \tag{16}$$

This does not represent a loss of generality, since any ΨDO has such an asymptotic expansion. The necessity of imposing $|p| \geq \delta$ arises from the fact that a homogeneous function is not, in general, smooth at the origin; in this sense, δ should be thought of as an infrared regulator to be taken to zero at the end.

Any asymptotic expansion (14) defines the symbol of a ΨDO up to an infinitely smoothing operator and we can therefore use the same equivalence sign "≈" between two asymptotic expansions. One way to convince oneself that this is true is to introduce a C^∞ function $\phi : \mathbb{R}_+ \to \mathbb{R}$ such that $\phi(t) = 0$ for $t < 1/2$ and $\phi(t) = 1$ for $t > 1$, and set

$$s(x,p) = \sum_{r \geq 0} \phi(|p|/(1+r)) s_{m-r}(x,p). \tag{17}$$

It can be proven that $s(x,p)$ is the symbol of a ΨDO of order m. Although there is a lot of arbitrariness in the choice of $s(x,p)$ it should be evident that two such symbols can only differ by an infinitely smoothing operator. Note that the regulating function ϕ in the series (17) for $s(x,p)$ above has the effect of truncating the series for any given value of $|p|$ to a finite number of terms, and that the number of terms grows with increasing $|p|$.

The basic operation in symbol calculus is the star product, corresponding to the (noncommutative) multiplication of operators on Hilbert space. In other words, the star product of the symbols of two operators S and R is defined as the symbol of the composite operator:

$$\mathrm{sym}\,(S) * \mathrm{sym}\,(R) \approx \mathrm{sym}\,(SR). \tag{18}$$

The asymptotic expansion of the star product of two symbols is

$$(s * r)(x,p) \approx \sum_{k=0}^{\infty} \frac{(-i)^k}{k!} \frac{\partial^k s}{\partial p_{\mu_1} \cdots \partial p_{\mu_k}} \frac{\partial^k r}{\partial x^{\mu_1} \cdots \partial x^{\mu_k}}, \tag{19}$$

and we may formally write

$$* = \exp(-i \frac{\overleftarrow{\partial}}{\partial p_\mu} \frac{\overrightarrow{\partial}}{\partial x^\mu}). \tag{20}$$

Note from the first term in the expansion (19) that $\mathrm{ord}\,(SR) = \mathrm{ord}\,(S) + \mathrm{ord}\,(R)$. Although we have not explicitly inserted one in (19), a regulator is needed if one, as we do here, wants to deal with smooth symbols only. Consequently, (19) defines such a smooth function only up to an infinitely smoothing operator.

The asymptotic behavior of the symbol also determines whether the corresponding operator is bounded, HS or trace class; in any dimension n, S is bounded iff $\mathrm{ord}\,(S) \leq 0$, HS iff $\mathrm{ord}\,(S) < -(n/2)$ and trace class iff $\mathrm{ord}\,(S) < -n$, the last two inequalities being in the strict sense. For a trace class ΨDO one could, of course, compute the trace as in def. (6), which by Fourier analysis would lead to

$$\mathrm{Tr}(S) = \int_{\mathbb{R}^n} \frac{d^n p}{(2\pi)^n} \int_D d^n x \, \mathrm{tr}\, s. \tag{21}$$

There are, however, a couple of problems with this expression. One is that it is not well defined on the equivalence classes of ΨDOs; for instance, $\mathrm{Tr} e^{-|\Delta|} \neq 0$. This means, for example, that one should be careful in using asymptotic expressions like (19) inside this trace. Another problem, which actually turns out to be a blessing in disguise, is that, if we fix some specific order for evaluating the integrals and the finite-dimensional trace tr, (21) gives a finite number for a much larger class of ΨDOs. For example, if we decide to take the finite-dimensional trace first, then

(21) will vanish for any symbol of the type $s(x,p) = f(x,p)T$, $f : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{C}$, $T \in \mathfrak{gl}(M,\mathbb{C})$ traceless, independently of the the asymptotic behavior of s. Thus, by choosing a particular order of integration, we can considerably enlarge the set of symbols yielding a finite answer. In [Mi-94] it was argued that the right order is to take the radial momentum integral, the only potentially divergent one, after the trace and all other integrals. The reason why this is the right thing to do will become abundantly clear from the calculations in sections 5, 6 and 7.

Quite independent of the above concepts is another trace that one can define on the space of ΨDOs. This trace, known as the Wodzicki residue [Wo-85, Gu-85, Ad-79, Kr-91, Ma-79], has many advantages over the one defined by (21). Thus, consider a ΨDO S with symbol s having an asymptotic expansion of the form (14). The Wodzicki residue of S is defined as

$$\mathrm{Res}\,(s) = \frac{1}{(2\pi)^n} \int_{D\times S^{n-1}} \mathrm{tr}\, s_{-n}(x,p)\eta(d\eta)^{n-1}, \tag{22}$$

where $\eta = p_\mu dx^\mu$ is the canonical one-form, S^{n-1} is the sphere $|p| = \delta$ in momentum space and we are assuming, as always, that s_{-n} is homogeneous for $|p| \geq \delta$. Note that (22) is independent on the radius of the sphere δ, as long as we assume s_{-n} to be homogeneous outside, and we could also consider the limit $\lim_{\delta\to 0^+}$ of (22) as a way of removing the infrared regulators. Since we are only considering flat space, expression (22) reads:

$$\mathrm{Res}\,(s) = \frac{\delta^n}{(2\pi)^n} \int_{|p|=\delta} d\Omega \int_D d^n x\; \mathrm{tr}\, s_{-n}(x,p), \tag{23}$$

$d\Omega$ being the angular integration over the sphere $|p| = \delta$. The residue is a linear functional operator defined on the space of ΨDO equivalence classes. Notice that it vanishes identically for trace class operators.

The Wodzicki residue can be used to construct a non-trivial two-cocycle on the Lie algebra of ΨDOs by

$$c_R(A,B) = \mathrm{Res}\,([\log|p|, a]_* * b), \tag{24}$$

where $a = \mathrm{sym}\, A$, $b = \mathrm{sym}\, B$. This so-called Radul cocycle [Ra-91a,b] defines a non-trivial central extension of the Lie algebra of ΨDOs. It also arises in applications of noncommutative differential geometry [Co-85, Co-88].

The reader should note that $\log|p|$ is really a singular function at the origin. However, the residue is a boundary integral and therefore independent of the way $\log|p|$ is regularized at the origin. We also would like to mention that $\log|p|$ does not have an asymptotic expansion in the sense of eq. (14). This does not cause any problem, however, since only its derivatives appear in the residue.

5. An important lemma: the Radul cocycle as a commutator

In this section we prove the following identity, to be used in sect. 7:

$$c_R(A,B) = -\int_{\mathbb{R}^n} \frac{d^n p}{(2\pi)^n} \int_D d^n x\, \mathrm{tr}([a,b]_*|_{-n}). \tag{25}$$

The importance of the order of the integrals on the r.h.s. is already clear at this stage and will become even more obvious after the calculations. The integrand is not the symbol of a trace class ΨDO ; if it were, being a commutator, its trace would vanish. However, by taking the integrals in the order indicated above, we will be able to prove that the r.h.s is well defined (i.e. independent of the regulators for the star product) and coincides with the Radul cocycle. Also, notice that the resemblance of equation (25) with a coboundary $\delta\lambda(A,B) = \lambda([A,B])$ is illusory; the apparent one-cochain

$$\lambda(A) = -\int_{\mathbb{R}^n} \frac{d^n p}{(2\pi)^n} \int_D d^n x \, \mathrm{tr} a|_{-n} \tag{26}$$

does not exist since the integral does not converge for a generic element A in the class of ΨDOs we are interested in (e.g. $a = (1+|p|)^{-n}$).

After these words of caution, let us turn to the proof. Consider two smooth symbols a and b, homogeneous of degree k_a and k_b for $|p| \geq \delta$. We prove eq. (25) for such symbols — the complete result follows from linearity.

Let $N = k_a + k_b + n$. This is the number of p-derivatives needed to reach a symbol of degree $-n$. Using eq. (19) for the star product, the integrand of the right hand side of eq. (25) is written

$$\mathrm{tr}\,[a,b]_*|_{-n} = \ \mathrm{tr}\, \frac{(-i)^N}{N!} \frac{\partial^N a}{\partial p_{\mu_1} \ldots \partial p_{\mu_N}} \frac{\partial^N b}{\partial x^{\mu_1} \ldots \partial x^{\mu_N}} - (a \leftrightarrow b). \tag{27}$$

No regulating function ϕ is needed in (27) because we are dealing with a finite sum of smooth functions. Integration by parts in x is always allowed since the symbols have compact support. We use this fact to move all x-derivatives to b and then identify the integrand as a total divergence:

$$\begin{aligned}
&\int_{\mathbb{R}^n} \frac{d^n p}{(2\pi)^n} \int_D d^n x \, \mathrm{tr}[a,b]_*|_{-n} \\
&\quad = \int_{\mathbb{R}^n} \frac{d^n p}{(2\pi)^n} \int_D d^n x \, \mathrm{tr} \frac{(-i)^N}{N!} \Bigg(\frac{\partial^N a}{\partial p_{\mu_1} \ldots \partial p_{\mu_N}} \frac{\partial^N b}{\partial x^{\mu_1} \ldots \partial x^{\mu_N}} \\
&\qquad\qquad\qquad\qquad - (-1)^N a \frac{\partial^{2N} b}{\partial p_{\mu_1} \ldots \partial p_{\mu_N} \partial x^{\mu_1} \ldots \partial x^{\mu_N}} \Bigg) \\
&\quad = \int_{\mathbb{R}^n} \frac{d^n p}{(2\pi)^n} \int_D d^n x \, \mathrm{tr} \frac{i^N}{N!} \frac{\partial}{\partial p_{\mu_1}} \sum_{m=0}^{N-1} (-1)^{m-1} \frac{\partial^m a}{\partial p_{\mu_2} \ldots \partial p_{\mu_{m+1}}} \frac{\partial^{2N-m-1} b}{\partial p_{\mu_{m+2}} \ldots \partial p_{\mu_N} \partial x^{\mu_1} \ldots \partial x^{\mu_N}} \\
&\quad \equiv \int_{\mathbb{R}^n} \frac{d^n p}{(2\pi)^n} \int_D d^n x \, \mathrm{tr} \frac{\partial}{\partial p_\mu} V_\mu.
\end{aligned} \tag{28}$$

For later comparison with the Radul cocycle, we have defined quantity

$$V_\mu = \frac{i^N}{N!} \, \mathrm{tr} \sum_{m=0}^{N-1} (-1)^{m-1} \frac{\partial^m a}{\partial p_{\mu_2} \ldots \partial p_{\mu_{m+1}}} \frac{\partial^{2N-m-1} b}{\partial p_{\mu_{m+2}} \ldots \partial p_{\mu_N} \partial x^{\mu} \partial x^{\mu_2} \ldots \partial x^{\mu_N}}, \tag{29}$$

which is homogeneous of degree $(-n+1)$ for $|p| \geq \delta$. Since the integrand is a total divergence it follows that the integral is scale invariant, i.e. independent of an ultraviolet cut-off. Thus, it can be written as a surface integral that may be pulled back from infinity to δ:

$$\int_{\mathbb{R}^n} \frac{d^n p}{(2\pi)^n} \int_D d^n x \, \mathrm{tr}[a,b]_*|_{-n} = \frac{\delta^{n-2}}{(2\pi)^n} \int_{|p|=\delta} d\Omega \int_D d^n x p_\mu V_\nu \delta^{\mu\nu}. \tag{30}$$

Let us now calculate the Radul cocycle explicitly. The integrand is

$$
\begin{aligned}
&\mathrm{tr}\left([\log|p|,a]_* * b\right)|_{-n} \\
&= \ \mathrm{tr}\sum_{q=1}^{N}\frac{(-i)^N}{q!(N-q)!}\frac{\partial^{N-q}}{\partial p_{\mu_{q+1}}\dots\partial p_{\mu_N}}\left(\frac{\partial^q \log|p|}{\partial p_{\mu_1}\dots\partial p_{\mu_q}}\frac{\partial^q a}{\partial x^{\mu_1}\dots\partial x^{\mu_q}}\right)\frac{\partial^{N-q}b}{\partial x^{\mu_{q+1}}\dots\partial x^{\mu_N}}.
\end{aligned}
\tag{31}
$$

Here we need to integrate by parts not only in x but also in p, which is allowed since

$$
\int_{D\times S^{n-1}}\left(\frac{\partial}{\partial p_\mu}w_\mu\right)\eta(d\eta)^{n-1} = 0 \tag{32}
$$

when w_μ is homogeneous of degree $(-n+1)$. This follows from the fact that (32) is the integral of an exact form

$$
\left(\frac{\partial}{\partial p_\mu}w_\mu\right)\eta(d\eta)^{n-1} = d\left(\frac{1}{n-1}w_\mu dx^\mu \eta(d\eta)^{n-2}\right) \tag{33}
$$

over a manifold with boundary $\partial(D\times S^{n-1})\equiv \partial D\times S^{n-1}$ where ω_μ vanishes due to the assumed spatial boundary conditions.

In one dimension the p-integral reduces to a sum over $S^0=\{\pm\delta\}$. Eq. (33) does not apply in this case, but since the derivative of a homogeneous function of degree zero vanishes, eq. (32) holds trivially and formal partial integration is allowed. We may thus shift all the p-derivatives except one from $\log|p|$ and use $\frac{\partial \log|p|}{\partial p_\mu} = \frac{\delta^{\mu\nu}p_\nu}{p^2}$ to obtain

$$
\begin{aligned}
&\frac{1}{(2\pi)^n}\int_{D\times S^{n-1}} \mathrm{tr}\left([\log|p|,a]_* * b\right)|_{-n}\eta(d\eta)^{n-1} \\
&\quad = \frac{1}{(2\pi)^n}\int_{D\times S^{n-1}}\eta(d\eta)^{n-1}\ \mathrm{tr}\, i^N\sum_{q=1}^{N}\sum_{m=0}^{q-1}\frac{(-1)^{q-1}}{q!(N-q)!}\binom{q-1}{m} \\
&\qquad\quad \times\ \frac{\delta^{\mu_1\mu}p_\mu}{p^2}\frac{\partial^m a}{\partial p_{\mu_2}\dots\partial p_{\mu_{m+1}}}\ \frac{\partial^{2N-m-1}b}{\partial p_{\mu_{m+2}}\dots\partial p_{\mu_N}\partial x^{\mu_1}\dots\partial x^{\mu_N}}.
\end{aligned}
\tag{34}
$$

Interchanging the order of summation and performing the sum over q,

$$
\sum_{q=m+1}^{N}(-1)^{q-1}\binom{N}{q}\binom{q-1}{m} = (-1)^m, \tag{35}
$$

brings the result

$$
\mathrm{Res}\left([\log|p|,a]_* * b\right) = -\frac{\delta^{n-2}}{(2\pi)^n}\int_{|p|=\delta}d\Omega\int_D d^nx p_\mu V_\nu \delta^{\mu\nu}, \tag{36}
$$

with V_μ as in (29). This proves the lemma. For later purposes, notice that one can even take the limit $\lim_{\delta\to0^+}$ in all the equations above, effectively removing the infrared cut-off from the picture.

Notice that any term in the asymptotic expansion of $[a,b]_*$ after tracing over $\mathfrak{gl}(M,\mathbb{C})$ and integrating over x can be written as a total derivative in p. Therefore, any term of degree less than $-n$ vanishes upon integration over p because of the good ultraviolet asymptotic behavior. For the term of degree $-n$, on the other hand, the integral becomes scale invariant instead of having the

naive logarithmic divergence. This we think is at the very heart of the nature of anomalies; they are neither genuinely ultraviolet nor infrared, but exactly what is in between.

6. The embedding of $\Psi\mathfrak{gl}_{\mathrm{res}}$ in $\mathfrak{gl}_{\mathrm{res}}$ in three dimensions

Having discussed the basic properties of ΨDOs, we are now in a position to describe the subalgebra of ΨDOs in $\mathfrak{gl}_{\mathrm{res}}$, which we denote by $\Psi\mathfrak{gl}_{\mathrm{res}}$. From now on, we shall work in three dimensions only but it should be clear how to generalize the results to an arbitrary number of dimensions. As mentioned in sect. 2 we consider the case of Weyl fermions with an extra $\mathfrak{su}(N)$ degree of freedom so that $\mathscr{H} = \mathrm{L}^2(\mathbb{R}^3) \otimes \mathbb{C}^2_{\mathrm{spin}} \otimes \mathbb{C}^N_{\mathrm{color}}$. As before, we shall restrict ourselves to symbols with compact support D in the variable x. The spin algebra is generated by the usual Pauli matrices σ^μ ($\mu = 1, 2, 3$ are space indices).

Since the energy of a free Weyl fermion is given by $E = -i\sigma^\mu \partial_\mu$ †, the symbol associated to the sign of the energy is $\mathrm{sym}\,(\mathrm{sign}(E)) = \frac{\sigma^\mu p_\mu}{|p|}$. As such, this symbol is singular and requires an infrared regularization. Even if we will never need it explicitly, one way to regularize a symbol of this kind is to introduce a function ϕ similar to the one used in sect. 4, except that now $\phi(t) = 0$ for $t < \delta/2$, $\phi(t) = 1$ for $t > \delta$, and to set:

$$\mathrm{sym}\,(\mathrm{sign}(E)) = \phi(|p|)\frac{\sigma^\mu p_\mu}{|p|} \equiv \varepsilon. \tag{37}$$

We can now look for the conditions under which a ΨDO describes an element of $\mathfrak{gl}_{\mathrm{res}}$, i.e. has a good second quantization. Let A be a ΨDO acting on $\mathscr{H}$ with symbol

$$a(x,p) \approx \sum_{k \le m} a_k(x,p), \tag{38}$$

From sect. 2, eq. (4), we must require that A be bounded and $[\mathrm{sign}(E), A]$ be HS. Specializing the considerations of sect. 4 to the $n = 3$ case, we must require for a first of all that $m = 0$ and secondly the HS condition

$$\mathrm{ord}\,([\varepsilon, a]_*) \le -2. \tag{39}$$

The most general symbol satisfying these requirements is given by the asymptotic expansion

$$a(x,p) \approx \sum_{k \le 0} a_k(x,p), \tag{40}$$

with

$$\begin{aligned} a_0(x,p) &= \alpha_0(x,p) + \tilde\alpha_0(x,p)\varepsilon, \\ a_{-1}(x,p) &= \frac{i}{2}\varepsilon\varepsilon^\mu \frac{\partial}{\partial x^\mu}(\alpha_0(x,p) + \tilde\alpha_0(x,p)\varepsilon) + \alpha_{-1}(x,p) + \tilde\alpha_{-1}(x,p)\varepsilon, \\ a_k(x,p) &\quad \text{arbitrary} \quad \text{for} \quad k \le -2, \end{aligned} \tag{41}$$

where the expression ε^μ denotes the derivative of the symbol ε with respect to p_μ and α_0, $\tilde\alpha_0$, α_{-1} $\tilde\alpha_{-1}$ are four smooth symbols, homogeneous of degree 0 and -1 and proportional to the identity matrix in spin space.

† Strictly speaking, E reverses the chirality of the spinor. However, for our purposes, we can assume the existence of a fixed isomorphism between the two chiralities.

To verify that (41) is the most general solution of (39), expand the commutator $[\varepsilon, a]_*$ and impose that the contributions of terms of degree 0 and -1 vanishes. This requires:

$$\begin{aligned}[\varepsilon, a_0] &= 0, \\ [\varepsilon, a_{-1}] &= i\varepsilon^\mu \frac{\partial}{\partial x^\mu} a_0,\end{aligned} \tag{42}$$

where the commutators in (42) are ordinary commutators and we are considering solutions for $|p| \geq \delta$. The first of (42) has solution $a_0(x,p) = \alpha_0(x,p) + \tilde{\alpha}_0(x,p)\varepsilon$ since the only 2×2 matrices that commute with ε are the identity and ε itself. Plugging this solution into the second of (42), we see that it determines only the component of a_{-1} that anticommutes with ε. If we write $a_{-1} = a^C_{-1} + a^A_{-1}$, for the commuting and anticommuting component respectively, we obtain: $a^A_{-1} = \frac{i}{2}\varepsilon\varepsilon^\mu \frac{\partial}{\partial x^\mu}(\alpha_0(x,p) + \tilde{\alpha}_0(x,p)\varepsilon)$, whereas the commuting part is given by the most general solution $a^C_{-1} = \alpha_{-1}(x,p) + \tilde{\alpha}_{-1}(x,p)\varepsilon$. There are no further requirements on the star commutator and, therefore, terms of order ≤ -2 are arbitrary. This completes the proof of (41). We will in fact never need the explicit solutions (41) but only use the properties (42); (41) being given for completeness only. A final remark to be made is that if we tried to solve the second of (42) for $|p| < \delta$ we would have encountered the problem that, in general, the equation is not integrable because of the presence of the regulator. However, these problems do not arise for $|p| \geq \delta$ where $\varepsilon \equiv \frac{\sigma^\mu p_\mu}{|p|}$, making the r.h.s. anticommuting with ε and allowing to solve for a^A_{-1}.

7. On the cohomology of the Schwinger term in 3+1 dimensions

In this section, we prove the other main result of our paper: The Schwinger term for operators in $\mathfrak{gl}_{\text{res}}$, represented by the cocycle (9) when restricted to the subalgebra of ΨDOs $\Psi\mathfrak{gl}_{\text{res}}$

$$c_S(A,B) = \int_{\mathbb{R}^3} \frac{d^3p}{(2\pi)^3} \int_D d^3x \, \mathrm{tr}([\varepsilon, a]_* * b), \tag{43}$$

is cohomologically equivalent to the "twisted" Radul cocycle, defined as

$$c_{TR}(A,B) \equiv c_R(\mathrm{sign}(E)A, B) = \mathrm{Res}\,([\log|p|, \varepsilon * a]_* * b) = \mathrm{Res}\,(\varepsilon * [\log|p|, a]_* * b). \tag{44}$$

Eq. (43) should be interpreted as the limit

$$\begin{aligned} c_S(A,B) = \lim_{\delta \to 0^+} &\int_{\mathbb{R}^3} \frac{d^3p}{(2\pi)^3} \int_D d^3x \, \mathrm{tr} \sum_{k=0,-1,-2,-3} ([\varepsilon, a]_* * b)|_k \\ &+ \int_{\mathbb{R}^3} \frac{d^3p}{(2\pi)^3} \int_D d^3x \, \mathrm{tr}([\varepsilon, a]_* * b)|_{\leq -4}, \end{aligned} \tag{45}$$

where δ is the infrared regulator introduced in sect. 4. As we will see below, there is no need for an ultraviolet cut-off because the potentially divergent terms will turn out to be zero. Also, we denote by $s|_{\leq -4}$ a smooth ΨDO (representative) with asymptotic expansion $\sum_{k \leq -4} s_k$.

The notion of the twisted Radul cocycle was first introduced in [Mi-94]. To check that c_{TR} really is a two-cocycle is straightforward and will not be done here (see for instance [Mi-94]). What is not obvious, however, is that, despite the fact that expression (45) is *not* well defined on the

equivalence classes of asymptotic expansions of ΨDOs because of the ambiguity of the integral in the presence of a regulator, its cohomology is still well defined in the sense that all dependence on the regularization can be written as an exact piece $\delta\lambda(A,B)$, the Lie algebra coboundary of a one-cochain λ to be specified below.

The equivalence between these two cocycles was shown to hold for a more restricted class of operators already in [Mi-94] and, subsequently, in [Fe-94] for another small class of operators. We show here that the equivalence is in fact true for all ΨDOs in $\mathfrak{gl}_{\rm res}$. All previous results follow straightforwardly from this one. Also, our proof keeps careful track of all the regulators and allows us to settle some unresolved issues in the previous literature.

What we will prove is that, for any two operators A and B in $\Psi\mathfrak{gl}_{\rm res}$ defined through their asymptotic expansions of the form given in eqs. (40) and (42), the following relation holds:

$$c_S(A,B) = \delta\lambda(A,B) + c_{TR}(A,B). \tag{46}$$

Here c_S and c_{TR} are defined as in (45) and (44) and

$$\lambda(A) = \lim_{\delta\to 0^+} \int_{|p|\le\delta} \frac{d^3p}{(2\pi)^3} \int_D d^3x\, \mathrm{tr}(\varepsilon * a)|_{-3} + \int_{\mathbb{R}^3} \frac{d^3p}{(2\pi)^3} \int_D d^3x\, \mathrm{tr}(\varepsilon * a)|_{\le -4}. \tag{47}$$

The proof proceeds as follows: Because of the associativity of the star product, the following relation between asymptotic expansions holds true:

$$[\varepsilon, a]_* * b \approx \varepsilon * [a,b]_* + [\varepsilon * b, a]_*. \tag{48}$$

Now consider the asymptotic expansion of the l.h.s. in terms of the asymptotic expansions of a and b. The terms of degree 0 and -1 do not appear because $[\mathrm{sign}(E), A]$ is a HS operator and B is bounded. The terms of degree -2 and -3 can readily be worked out:

$$\begin{aligned}
[\varepsilon,a]_* * b|_{-2} &= \big([\varepsilon, a_{-2}] - i\varepsilon^\mu \frac{\partial}{\partial x^\mu} a_{-1} - \frac{1}{2}\varepsilon^{\mu\nu}\frac{\partial^2}{\partial x^\mu \partial x^\nu} a_0\big) b_0,\\
[\varepsilon,a]_* * b|_{-3} &= \big([\varepsilon, a_{-3}] - i\varepsilon^\mu \frac{\partial}{\partial x^\mu} a_{-2} - \frac{1}{2}\varepsilon^{\mu\nu}\frac{\partial^2}{\partial x^\mu \partial x^\nu} a_{-1} + \frac{i}{6}\varepsilon^{\mu\nu\rho}\frac{\partial^3}{\partial x^\mu \partial x^\nu \partial x^\rho} a_0\big) b_0\\
&\quad + \big([\varepsilon, a_{-2}] - i\varepsilon^\mu \frac{\partial}{\partial x^\mu} a_{-1} - \frac{1}{2}\varepsilon^{\mu\nu}\frac{\partial^2}{\partial x^\mu \partial x^\nu} a_0\big) b_{-1}\\
&\quad - i\frac{\partial}{\partial p_\rho}\big([\varepsilon, a_{-2}] - i\varepsilon^\mu \frac{\partial}{\partial x^\mu} a_{-1} - \frac{1}{2}\varepsilon^{\mu\nu}\frac{\partial^2}{\partial x^\mu \partial x^\nu} a_0\big)\frac{\partial}{\partial x^\rho} b_0.
\end{aligned} \tag{49}$$

Consider the function of p arising by taking the finite-dimensional trace and the integral over the compact domain D of x for both terms in the expansion (49):

$$\begin{aligned}
F_{-2}(p) &= \int_D d^3x\ \mathrm{tr}\,[\varepsilon, a]_* * b|_{-2},\\
F_{-3}(p) &= \int_D d^3x\ \mathrm{tr}\,[\varepsilon, a]_* * b|_{-3}.
\end{aligned} \tag{50}$$

The crucial fact is that these two functions *vanish* outside the sphere $|p| = \delta$. For example, in the case of F_{-2}, one can check that neither a_{-2} nor a^C_{-1} survives the finite-dimensional trace and that the remaining terms combine to

$$F_{-2}(p) = \frac{1}{2}\int_D d^3x\ \mathrm{tr}\,(\varepsilon^\mu \varepsilon \varepsilon^\nu - \varepsilon^{\mu\nu}) a_0 \frac{\partial^2}{\partial x^\mu \partial x^\nu} b_0, \tag{51}$$

which is zero for $|p| \geq \delta$ because of the identity

$$\varepsilon^{\mu\nu} + \varepsilon\varepsilon^{\mu\nu}\varepsilon + \varepsilon^{\mu}\varepsilon^{\nu}\varepsilon + \varepsilon^{\nu}\varepsilon^{\mu}\varepsilon = 0, \tag{52}$$

following by taking two p derivatives of $\varepsilon \equiv \varepsilon\varepsilon\varepsilon$. In a similar way, the reader can check that also $F_{-3}(p)$ vanishes for $|p| \geq \delta$.

Using these results, one can restrict the integration over p to the region $|p| \leq \delta$ for the first four terms ($k = 0$, -1, -2 and -3) in the asymptotic expansion. (Obviously, the behavior of these functions inside the sphere depends on the regulator.) Now note that the integral over $|p| \leq \delta$ of any smooth symbol of degree $k = 0$, -1 or -2 vanishes as we let δ go to zero:

$$\lim_{\delta\to 0^+} \int_{|p|\leq\delta} \frac{d^3p}{(2\pi)^3} \int_D d^3x\, \mathrm{tr}(s_k) = 0 \quad \text{for} \quad k = 0, -1, -2. \tag{53}$$

Using this fact in (45), one can therefore write:

$$\begin{aligned} c_S(A,B) = & \lim_{\delta\to 0^+} \int_{|p|\leq\delta} \frac{d^3p}{(2\pi)^3} \int_D d^3x\, \mathrm{tr}([\varepsilon, a]_* * b|_{-3}) \\ & + \int_{\mathbb{R}^3} \frac{d^3p}{(2\pi)^3} \int_D d^3x\, \mathrm{tr}([\varepsilon, a]_* * b|_{\leq -4}). \end{aligned} \tag{54}$$

Using the property (53) also on the r.h.s. of (48), and comparing with the definition (47) we obtain:

$$\begin{aligned} c_S(A,B) = \delta\lambda(A,B) + & \lim_{\delta\to 0^+} \int_{|p|\leq\delta} \frac{d^3p}{(2\pi)^3} \int_D d^3x\, \mathrm{tr}([\varepsilon * b, a]_*|_{-3}) \\ & + \int_{\mathbb{R}^3} \frac{d^3p}{(2\pi)^3} \int_D d^3x\, \mathrm{tr}([\varepsilon * b, a]_*|_{\leq -4}. \end{aligned} \tag{55}$$

Eq. (46) then follows directly from the results proven in the sect. 5:

$$\lim_{\delta\to 0^+} \int_{|p|\leq\delta} \frac{d^3p}{(2\pi)^3} \int_D d^3x\, \mathrm{tr}([\varepsilon * b, a]_*|_{-3}) = -c_R(\mathrm{sign}(E)B, A) \equiv c_{TR}(A,B) \tag{56}$$

and

$$\int_{\mathbb{R}^3} \frac{d^3p}{(2\pi)^3} \int_D d^3x\, \mathrm{tr}([\varepsilon * b, a]_*|_{\leq -4}) = 0. \tag{57}$$

This completes the proof of (46). We remark once again that its importance relies not only on the fact that it relates two seemingly independent cocycles for the whole space $\Psi\mathfrak{gl}_{\mathrm{res}}$ but also in the fact that it shows that the cohomology of the Schwinger term is well defined in terms of ΨDOs, all the dependence on the regulators being swept into a coboundary.

8. Conclusions

In this paper we have shown how to relate two seemingly unrelated concepts such as the Schwinger term arising in second quantization and the Radul cocycle. There are numerous applications, some of which have already appeared in the literature, that relate directly to our general theorem. We simply quote some of them. To begin with, one can indeed reproduce the extension

arising in affine Kac–Moody algebras from the quantization of maps from S^1 to a simple Lie algebra [Ka-85] and in fact generalize these results from multiplicative operators to ΨDOs. Even more interesting is the three-dimensional case, which we have discussed at length. If one uses ΨDOs as regularizing counterterms for higher-dimensional current algebras, as recently proposed by Mickelsson [Mi-93, Mi-94], one can reproduce the extension arising in the gauge commutation relations for anomalous chiral gauge theories directly from the normal-ordered regulated gauge transformations. Other higher-dimensional current algebras, like the one proposed by us [Ce-94] also admit such a regularization [Fe-94]. Work is in progress in trying to understand the possible representation theory for these algebras and we hope to return on the subject in a later publication.

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